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CLASSIFYING VERTEX-TRANSITIVE GRAPHS WHOSE ORDER IS A PRODUCT OF TWO PRIMES

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Vertex-transitive graphs whose order is a product of two primes with a primitive automorphism group containing no imprimitive subgroup are classified. Combined with the results of [15] a complete classification of all vertex-transitive graphs whose order is a product of two primes is thus obtained (Theorem 2.1).

1. Introduction

By p and q we shall denote distinct prime numbers, p being the larger one. Unless otherwise specified the graphs considered are finite, simple and undirected. By an *n-graph* we shall always mean a graph with n vertices.

In this paper we complete the classification of vertex-transitive pq-graphs initiated in [15] where imprimitive graphs – that is those graphs among them which have an imprimitive subgroup of automorphisms – are characterized (see Proposition 1.1 below).

If X is a vertex-transitive pq-graph having an imprimitive subgroup of $AutX$ with blocks of size p, then it has to be a metacirculant [12]. (Note that vertex-transitive p^2-graphs are Cayley graphs of Abelian groups and therefore metacirculants.) Metacirculants were introduced by Alspach and Parsons [1] who also gave an algebraic characterization for these graphs. An (m,n)-*metacirculant* has an automorphism f with a cycle decomposition

$$f = (v_0^0\, v_1^0 ... v_{n-1}^0)(v_0^1\, v_1^1 ... v_{n-1}^1)...(v_0^{m-1}\, v_1^{m-1} ... v_{n-1}^{m-1})$$

and an automorphism g which cyclically permutes the orbits

$$V_i = \{v_0^i, v_1^i, ..., v_{n-1}^i\}\ (i = 0, 1, ..., m-1).$$

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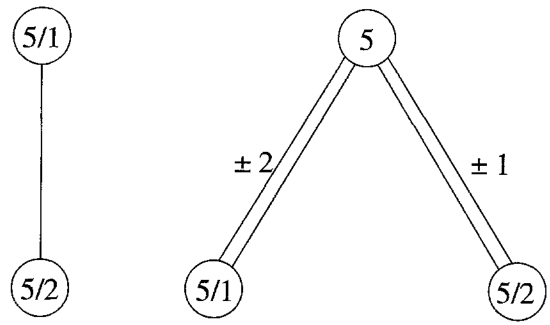

Fig. 1. The Petersen graph and its line graph.

It can be shown that there exists $r \in Z_m^*$ such that $g(v_j^i) = v_{rj}^{i+1}$ for all i and j, forcing $r^h \equiv 1\,(mod\,n)$ for some multiple h of m. In short, a metacirculant is a graph with a transitive cyclic or metacyclic subgroup. In the first case the term *circulant* is used. It is shown in [1] that every (m,n)-metacirculant can be associated with an array $(m,n,r,S_0,S_1,...,S_k)$ where $k = \lfloor \frac{m}{2} \rfloor$, $S_i = \{s \in Z_n : v_0^0 v_s^i$ is an edge$\}$ and $r^m S_i = S_i$. Let $M(m,n,r,S_0,S_1,...,S_k)$ denote the graph associated with this array.

As for vertex-transitive pq-graphs having imprimitive subgroups of automorphisms with blocks of size q – other than circulants – they are rather rare. In fact, they only exist when p is a Fermat prime and q divides $p-2$. They arise from an imprimitive action of $SL(2,p-1)$ with $PG(1,p-1)$ as the complete block system and are described in the following way [14, 15]. Let w be a fixed generator of $GF(p-1)^*$. For a symmetric subset S of $GF(q)^*$ and a non-empty proper subset T of $GF(q)^*$ we let $F(p,q,S,T)$ denote the graph with vertex-set $PG(1,p-1) \times GF(q)$ such that, for each $v \in PG(1,p-1)$ and each $r \in GF(q)^*$, the neighbors of (v,r) are all the vertices of the form $(\infty, r+s), (s \in S)$ and $(y, r+t), (y \in GF(p-1), t \in T)$ if $v = \infty$ and all the vertices of the form $(v, r+s), (s \in S)$ and $(\infty, r-s), (t \in T)$ and $(v+w^i, -r+t+2i), (i \in GF(q), t \in T)$ if $v \neq 0$. We let $F(p,q,S,T)$ be called a *Fermat graph.* Using Frucht's notation [6], Figure 1 gives the Petersen graph – the smallest metacirculant which is not a circulant – and its line graph which is the smallest Fermat graph.

In view of the above comments the following result characterizes vertex-transitive pq-graphs with an imprimitive subgroup of automorphisms.

Proposition 1.1. [15, Theorem] *Let X be a pq-graph. Then $AutX$ has an imprimitive subgroup of automorphisms if and only if either*
(i) *there exist an element $r \in GF(q)^*$ satisfying $r^q \equiv 1\,(mod\,p)$, a symmetric subset S_0 of $GF(p)^*$ and subsets $S_i\,(i = 1,2,,...,\frac{q-1}{2})$ such that X is isomorphic to the metacirculant $M(q,p,r,S_0,S_1,...S_{\frac{q-1}{2}})$ or*
(ii) *there exist a symmetric subset S of $GF(q)^*$ and a subset T of $GF(q)$ such that X is isomorphic to the Fermat graph $F(p,q,S,T)$.*

Moreover the classes of Fermat graphs and metacirculants are disjoint. ∎

Now, to complete the classification of all vertex-transitive pq-graphs a thorough analysis of all vertex-transitive pq-graphs with primitive automorphism groups, all of whose transitive subgroups are also primitive, is needed. Such graphs will be called *primitive*. This terminology may be a bit non-standard, but we think that the description of vertex-transitive pq-graphs with a primitive automorphism group containing an imprimitive subgroup is more naturally obtained via Proposition 1.1. (In particular, in our terminology a primitive graph with a non-prime number of vertices is necessarily a non-Cayley graph.) Primitive pq-graphs are thus obtained from *uniprimitive groups* (that is primitive but not doubly transitive groups) of degree pq without imprimitive subgroups via the standard orbital graph construction described below (see also [2]).

Let G be a transitive permutation group on a set V. To an element $v \in V$ and a suborbit W of G relative to v – that is an orbit of the stabilizer G_v of v in G – we associate a binary relation R_W on V where xR_Wy if and only if there exists $g \in G$ such that $g(x) = v$ and $g(y) \in W$. If W is non-trivial ($\neq \{v\}$) we let the *orbital graph* $X(G,W)$ *of* G *with respect to* W be the graph with vertex-set V and arc-set R_W. Of course, $X(G,W)$ is undirected if and only if W is self-paired, that means when R_W is a symmetric relation on V. Taking this definition a little further, a *generalized orbital graph* $X(G,W)$ of G with respect to a non-empty, self-paired union $W \neq V \setminus \{v\}$ of suborbits of G has vertex-set V and arc-set R_W. A particular example of this situation is given by a group G acting faithfully on the set $\mathcal{H}$ of right cosets of a subgroup H by right multiplication. Let $\mathcal{W} \neq \mathcal{H} \setminus \{H\}$ be a non-empty self-paired union of non-trivial suborbits of G associated with this action and let $R_{\mathcal{W}}$ be the union of the corresponding binary relations on $\mathcal{H}$. To specify the group H we use the notation $X(G,H,\mathcal{W})$ for the corresponding generalized orbital graph.

It is hoped that the classification given here may prove useful in partially resolving the Lovász's problem on the existence of Hamilton paths in connected vertex-transitive graphs [11]. For example, the Petersen graph is the only non-hamiltonian graph among vertex-transitive pq-graphs with an imprimitive subgroup of automorphisms (see [13]). Some current work on hamiltonian properties of primitive pq-graphs will be the content of a sequel to this paper.

Finally, let us remark that it came to our notice while revising this paper that a classification of arc-transitive pq-graphs - a subclass of vertex-transitive pq-graphs - has been obtained by Praeger, Wang and Xu [19]. Rather than describing their graph-theoretical structure, the classification is done in terms of the whole automorphism group.

2. The classification

The main tool in our classification is the Liebeck-Saxl's list of all non-abelian socles of primitive groups of degree mp, where $m < p$ [10, Table 3, p. 239]. Note that if such a primitive group G has no imprimitive subgroups, its socle cannot be abelian. Namely, if this was the case the socle of G would be transitive and therefore regular. This however is impossible since mp is not a prime power. Therefore the automorphism group of a primitive pq-graph must necessarily contain as a subgroup a group in the above mentioned list. Besides, the extra conditions that the group is not doubly transitive and that $m=q$ is a prime, reduces this list to the one given in Table 1. The arithmetic arguments are in most cases quite straightforward, in some less obvious.

Table 1. Non-abelian socles of uniprimitive groups of degree pq

row	*soc G*	(p,q)	*action*	*comment*
1	A_p	$(p,\frac{p-1}{2})$	pairs	$p\geq 5$
2	A_{p+1}	$(p,\frac{p+1}{2})$	pairs	$p\geq 5$
3	A_7	$(7,5)$	triples	
4	$PSL(4,2)$	$(7,5)$	2-spaces	
5	$PSL(5,2)$	$(31,5)$	2-spaces	
6	$PSp(4,k)$	$(k^2+1,k+1)$	1-spaces	p and q Fermat primes
7	$P\Omega^{\varepsilon}(2d,2)$	$(2^d-\varepsilon,2^{d-1}+\varepsilon)$	singular 1-spaces	$\varepsilon=+1$: d Fermat prime $\varepsilon=-1$: $d-1$ Mersenne prime
8	$PSL(2,p)$	$(p,\frac{p+1}{2})$	cosets of D_{p-1}	$p\equiv 1\,(mod\,4)$ $p\geq 13$
9	$PSL(2,p)$	$(p,\frac{p-1}{2})$	cosets of D_{p+1}	$p\equiv 3\,(mod\,4)$
10	$PSL(2,q^2)$	$(\frac{q^2+1}{2},q)$	cosets of $PGL(2,q)$	
11	$PSL(2,p)$	$(19,3)$, $(29,7)$ $(59,29)$, $(61,31)$	cosets of A_5	
12	$PSL(2,23)$	$(23,11)$	cosets of S_4	
13	$PSL(2,11)$	$(11,5)$	cosets of A_4	
14	M_{23}	$(23,11)$	see [4]	
15	M_{22}	$(11,7)$	see [4]	
16	M_{11}	$(11,5)$	see [4]	

Such is the action of $PSL(d,k)$ on 2-subspaces of the above mentioned Liebeck-Saxl's list. Its degree is $\frac{(k^d-1)(k^{d-1}-1)}{(k^2-1)(k-1)}$. For this number to be a product of two primes in case $d=2s$ is even, we must have $(p,q)=(\frac{k^{d-1}-1}{k-1},\frac{k^d-1}{k^2-1})$, where both s and $2s-1$ are primes. Similarly, if $d=2s+1$ is odd, then we must have $(p,q)=(\frac{k^d-1}{k-1},\frac{k^{d-1}-1}{k^2-1})$, with both s and $2s+1$ being primes. These conditions imply that $s=2$. Namely, for $s>2$ it follows that q has the form $1+k^2+\ldots.+k^{4j}$, with $j\geq 1$. Thus letting $a=1+k^2+\ldots.+k^{2j}$ and $b=k+k^3+\ldots.+k^{2j-1}$ we see that $q=a^2-b^2$ is not a prime. Therefore $q=1+k^2$ must be a Fermat prime. Also, in this case $p\in$

$\{1+k+k^2, 1+k+k^2+k^3+k^4\}$. For $q>5$ we have $k \equiv 1 (mod 3)$ and $k \equiv 1 (mod 5)$ and so p would not be a prime. Hence $q=5$ giving us $PSL(4,2)$ and $PSL(5,2)$ as the only admissible groups in this case (see rows 4 and 5 of Table 1).

Similarly, for the degree $\frac{(k^d-\varepsilon)(k^{d-1}+\varepsilon)}{k-1}$ of the group $P\Omega^{\varepsilon}(2d,k), \varepsilon=\pm 1$, acting on singular 1-spaces (note that the group $P\Omega^{-}(2d,2)$ is missing in the Liebeck-Saxl's list) to be a product of two primes, we must have $k=2$. To illustrate the argument let $\varepsilon=-1$. Then we must have that $k=2^{2^r}$ and that $d=2^s$ for $r \geq 0$ and $s \geq 2$. If $k>2$ then $r \geq 1$ and so $k=h^2$ for some integer h. Since $d-1 \geq 3$ is odd we have that $q=\frac{k^{d-1}-1}{k-1}=\left(\frac{h^{d-1}-1}{h-1}\frac{h^{d-1}+1}{h+1}\right)$ is a composite number, a contradiction. So $k=2$ and therefore $q=2^{d-1}-1$. The primeness of q then implies that $d-1=2^s-1$ is a prime and thus s is a prime too. This justifies row 7 of Table 1.

The class of primitive pq-graphs is obtained following these steps. First, all those groups in Table 1 which have imprimitive subgroups are excluded. Next, it is proved that for the remaining groups the corresponding generalized orbital graphs are in fact primitive. Finally, it is made sure that those graphs among them which arise from more than one group are only counted once. We thus extract a reduced list of groups shown in Table 2. The proof that this list is complete, together with a detailed analysis of the various cases and the description of the relative generalized orbital graphs, is the main object of this paper and the content of Theorem 2.1 below.

Theorem 2.1. *A vertex-transitive pq-graph must be one of the following:*
(i) *a metacirculant,*
(ii) *a Fermat graph,*
(iii) *an orbital graph arising from certain rank 3 representations of* $P\Omega^{\pm}(2d,2)$ *or* M_{22},
(iv) *a generalized orbital graph associated with the action of* A_7 *on triples,*
(v) *a generalized orbital graph associated with a 2-dimensional projective special linear group.*

Furthermore the five classes of graphs above are mutually disjoint. Additional information is provided in Proposition 1.1 above for imprimitive graphs (i)-(ii) and in Table 2 for groups admitting primitive graphs (iii)-(v). A detailed description of these graphs is given in Sections 3, 4 and 5. ∎

The following simple group-theoretic result will be used in the proof of this theorem.

Lemma 2.2. *Let* G *be a transitive group on a set* V *of degree* n *and order* ns. *Let* H *be a subgroup of* G *such that* n *divides* $|H|$ *and* $([G:H],|H|)=1$. *Then* H *is transitive.*

Proof. Let $|H|=nt$. Then $(nt,\frac{s}{t})=1$. Let xH denote the orbit of H containing x. For any $x \in H$, we must have $|xH||H_x|=|H|$. Now $|H_x|$ divides $|H|$ as well as s since $H_x \subseteq G_x$. So $|H_x|$ divides (s,nt), which is equal to t. But then $|xH| \geq \frac{nt}{t}=n$. Thus $|xH|=n$. ∎

Proof of Theorem 2.1 . We shall first show that the groups in Table 1 not appearing in Table 2 either contain imprimitive subgroups and so the corresponding

Table 2. Uniprimitive groups of degree pq without imprimitive subgroups and with non-isomorphic generalized orbital graphs

soc G	(p,q)	*action*	*comment*	*case*
$P\Omega^{\varepsilon}(2d,2)$	$(2^d-\varepsilon, 2^{d-1}+\varepsilon)$	singular 1-spaces	$\varepsilon=+1$: Fermat prime $\varepsilon=-1$: $d-1$ Mersenne prime	(iii)
M_{22}	$(11,7)$	see [4]		(iii)
A_7	$(7,5)$	triples		(iv)
$PSL(2,p)$	$(p,\frac{p+1}{2})$	cosets of D_{p-1}	$p\equiv 1\,(mod\,4)$ $p\geq 13$	(v)
$PSL(2,q^2)$	$(\frac{q^2+1}{2},q)$	cosets of $PGL(2,q)$	$q\geq 5$	(v)
$PSL(2,61)$	$(61,31)$	cosets of A_5		(v)

generalized orbital graphs are metacirculants and Fermat graphs or give rise to generalized orbital graphs associated with some groups in Table 2.

Metacirculants arise precisely in these cases: from groups $PSL(2,19)$, $PSL(2,29)$, $PSL(2,59)$ in row 11 – in this proof row j will always stand for row j of Table 1 – and from all the groups in rows 1, 5, 9, 12, 13, 14 and 16. To see this we first show that in each of these cases the normalizer of a Sylow p-subgroup acts transitively and hence imprimitively.

Consider first the group A_p acting on pairs. If f denotes the p-cycle $(0\,1\,...\,p{-}1)$ and g is given by the formula $g(i)=ri\,(i\in GF(p))$, where r has order $\frac{p-1}{2}$ in $GF(p)^*$, then the normalizer of $<f>$ in A_p is $<f,g>$ which is easily seen to be transitive on pairs.

In all other cases the transitivity of the normalizer follows by Lemma 2.2. With the exception of the group $PSL(2,19)$ in row 11, all these normalizers contain a regular subgroup and so the corresponding graphs are Cayley. On the other hand, the generalized orbital graphs of the rank 4 group $PSL(2,19)$ are non-Cayley. In fact, apart from the Petersen graph and its complement these are the only non-Cayley metacirculants with a primitive automorphism group. (The one with smallest degree 6 is the distance-transitive Perkel graph [3].)

Moreover, none of the graphs in the remaining cases of Table 1 is a metacirculant. Namely, we have $p\not\equiv 1\,(mod\,q)$ and so by [1] to be a metacirculant such a graph would have to be a circulant, that is a Cayley graph of a cyclic group. However, by [22, Theorem 25.4] no simply primitive group contains a regular cyclic group of composite order.

Moving onto Fermat graphs, the necessary condition is of course that p is a Fermat prime and that q divides $p{-}2$. This is only satisfied for the rank 3 action of A_6 on pairs in row 2 as well as the group $PSL(2,9)$ – which is isomorphic to A_6 –

in row 10 and for the rank 3 action of the group $PSp(4,k)$ in row 6 with $p=k^2+k$ and $q=k+1$ being Fermat primes. Surprisingly the two orbital graphs in row 6 are in fact Fermat graphs. Namely, the group $PSp(4,k)$ contains an isomorphic copy of $PSL(2,k^2)$ acting imprimitively on 1-spaces. This follows in view of the fact that

$$SL(2,k^2) \subseteq Sp(4,k) : \text{for all prime powers } k. \tag{1}$$

To see this let I denote the identity 2×2-matrix, E be the 4×4-matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, $\mathcal{K}= GF(k)$ and $\mathcal{V}$ a 4-dimensional vector space of columns over $\mathcal{K}$. Now a symplectic form over $\mathcal{V}$ is given by $(X,Y)=X^tEY$. The group $Sp(4,k)$ then consists precisely of the 4×4-matrices S such that

$$S^tES = E. \tag{2}$$

Let $f(x) = x^2 + rx + s$ be an irreducible polynomial over K. Then $f(x)$ is the characteristic polynomial of the matrix $M = \begin{bmatrix} -r & -s \\ 1 & 0 \end{bmatrix}$. By the Hamilton-Cayley theorem $f(M)=0$. Let $\mathcal{K}'$ be the field of order k which consists of all 2-dimensional scalar matrices aI, where $a\in\mathcal{K}$. Then $M\notin\mathcal{K}'$ and $[\mathcal{K}'(M):\mathcal{K}']=2$. We may thus consider $SL(2,k^2)$ as the group of the 2×2-matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, $D\in \mathcal{K}'(M)$ and $AD-BC=I$, the unit of $\mathcal{K}'$. It is then not difficult to see that these matrices satisfy (2) obtaining thus an embedding of $SL(2,k^2)$ inside $Sp(4,k)$ and proving (1). A short computation shows that this group is transitive and — since it does not appear in the list of Liebeck and Saxl [10] — imprimitive on the 1-spaces of $\mathcal{V}$. This gives us an indirect proof of the fact that the two orbital graphs in row 6 are Fermat graphs, as they certainly cannot be metacirculants. Also, the strong regularity array of the one with smaller degree is $(k^3+k^2+k+1, k^2+k, k-1, k+1)$ (see [7]). Now it is not difficult to check that $F(p,q,GF(q)^*,\{0\})$ is the only Fermat graph with the same array, giving us a rather neat description of the two rank 3 graphs in row 6. In the smallest admissible case $(p,q) = (5,3)$ the two graphs are $L(K_6)$ and $L(K_6)^c$ which also turn out to be the orbital graphs for the group $A_6 = PSL(2,9)$ in rows 2 and 10. In Figure 2 the graph $L(K_6)^c$ is shown using the notation of Frucht [6] and emphasizing the three orbits of an automorphism of order 5.

We are now left with primitive groups without imprimitive subgroups. To conclude the proof we have to consider all those cases in which two different groups give rise to the same generalized orbital graph. Of course, this situation can only occur when the degrees of actions are equal. Using this fact, some tedious computations reduce the list of possible overlappings to three pairs of groups corresponding respectively to: rows 2 and 8, rows 3 and 4, and for $p = 61$ rows 2, 8 and 11.

The rank 3 action of A_{p+1} on pairs of $\{1,2,...,p+1\}$ contains the action of $PSL(2,p)$ on cosets of D_{p-1}. This is seen by identifying $\{1,2,...,p+1\}$ with the projective line over $GF(p)$ and observing that D_{p-1} is the stabilizer of a pair of points. As a consequence, the two orbital graphs associated with this action of

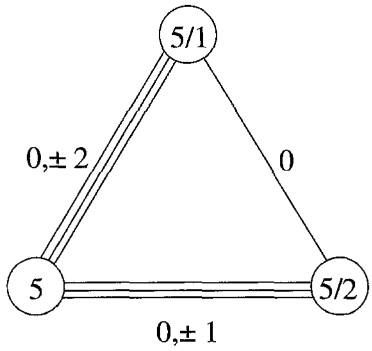

Fig. 2. $L(K_6)^c$.

A_{p+1} can be found among the generalized orbital graphs of the relative action of $PSL(2,p)$. This justifies the omission of row 2 of Table 1 from Table 2.

Next, as $PSL(4,2)$ is isomorphic to A_8, it certainly contains a copy of A_7 as a subgroup. It follows from [8, Satz 2.6, p. 157] that every A_7 inside $PSL(4,2)$ acts doubly transitively on the 15 non-zero vectors in $V(4,2)$ and so it acts transitively on 2-subspaces. This gives a transitive representation of A_7 of degree 35. But A_7 has only one conjugacy class of maximal subgroups of index 35 and so this representation is equivalent to the action of A_7 on triples. Note also that the groups $P\Omega^+(6,2)$ and $PSL(4,2)$ are isomorphic and the action of $P\Omega^+(6,2)$ on singular 1-spaces in row 7 of Table 1 is equivalent to the above action of $PSL(4,2)$. This justifies the additional condition $d \geq 5$ in row 1 of Table 2.

Let us now consider the possible overlapping of rows 2 and 11 for $p=61$. There is a unique permutation representation of $PSL(2,61)$ of degree 62, the natural action on the projective line. Also, the normalizer of this group in S_{62} contains an odd element, for example the one corresponding to the matrix $\begin{bmatrix} 1 & -5 \\ 0 & 11 \end{bmatrix}$. Hence all subgroups of A_{62} isomorphic to $PSL(2,61)$ are conjugate in A_{62}. Therefore considering the action of A_{62} on pairs, the subgroups isomorphic to $PSL(2,61)$ inside A_{62} give rise to an action on cosets of D_{60}. This shows that rows 2 and 11 give rise to disjoint classes of graphs.

Finally, suppose that there is a graph X which is a generalized orbital graph of $H = PSL(2,61)$ acting on cosets of D_{60} as well as of $H' = PSL(2,61)$ acting on cosets of A_5. Let $G = <H, H'>$. The socle S of G is a primitive group which must appear in the Liebeck-Saxl's list [10, Table 3, p. 239]. But apart from the two actions of $PSL(2,61)$, the only other such group is A_{62} acting on pairs. However, the previous argument excludes the embedding of H' in A_{62}. Thus rows 8 and 11 give rise to disjoint classes of graphs. ∎

The rest of the paper is devoted to the description of primitive graphs arising from groups in Table 2. 2.1 are dealt with in the next section. Section 4 gives the description of generalized orbital graphs associated with the action of $PSL(2,p)$ on the cosets of dihedral subgroups D_{p-1} and those associated with the action of $PSL(2,61)$ on cosets of A_5. Section 5 discusses the generalized orbital graphs arising from the action of $PSL(2,q^2)$ on the cosets of $PGL(2,q)$.

3. Low rank groups

In this section we discuss cases (iii) and (iv) of Theorem 2.1, that is the generalized orbital graphs associated with groups in rows 1, 2 and 3 of Table 2. With the exception of four graphs arising from the action of A_7 on triples, the rest are all rank 3 graphs. The graphs in Table 3 and their complements are precisely all of these graphs. (For the arithmetic conditions on p and q see Table 2).

Table 3. Generalized orbital graphs of low rank groups

group	vertex set	adjacency	degree
$PSL(4,2)$	2-spaces	intersecting spaces	18
$P\Omega^{\pm 1}(2d,2)$	singular 1-spaces	orthogonal spaces	$2(2^{d-1}\mp 1)(2^{d-2}\pm 1)$
M_{22}	blocks of $S(3,6,22)$	disjoint blocks	16
A_7	triples	empty intersection	4
A_7	triples	2-element intersection	12

The rank 3 graphs from the first three rows correspond to cases C1, C4 and S2 in the Hubaut's list [7, pp. 366-373]. A straightforward computation shows that the action of A_7 on the set of triples of $\{1,2,...,7\}$ gives rise to three orbital graphs of degrees 4, 12 and 18, with the respective adjacencies defined according to whether the intersection of two triples has no element, two elements or one element in common. This justifies the last two rows in Table 3. Of course, the graph with degree 18 is in fact the rank 3 graph from the first row of Table 3 (see also the proof of Theorem 2.1).

Note that the smallest admissible pair (p,q) of prime numbers for the second row in Table 3 is (17,7). Let us also mention that the rank 3 graphs in Table 3 and their complements together with the orbital graphs of the action of A_{p+1} on pairs – which correspond to case K2 in the above mentioned list – exhaust the class of all rank 3 primitive pq-graphs.

To end this section, observe that the graph in row 4 of Table 3 is the so called odd graph O_4. A neat representation of this graph may be given using the notation of Frucht [6] which emphasizes the five orbits $\{f^i(135): i=0,1,...,6\}$, $\{f^i(467): i=0,1,...,6\}$, $\{f^i(125): i=0,1,...,6\}$, $\{f^i(346): i=0,1,...,6\}$ and $\{f^i(127): i=0,1,...,6\}$ of the cycle (1234567) (see Figure 3 below).

4. Actions of $PSL(2,p)$

The generalized orbital pq-graphs arising from the action of projective linear special groups are classified in this and in the next section, by giving an explicit

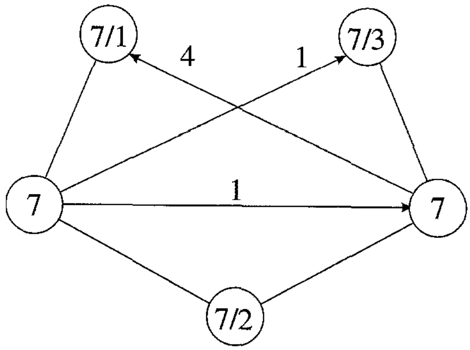

Fig. 3. Odd graph O_4.

description of suborbits. In particular, we compute the subdegrees of the corresponding actions. Let us also remark that subdegrees of all primitive permutation representations of $PSL(2,k)$ were calculated in [21]. This thesis is quite nonavailable, but some extractions appeared in [5].

The graphs arising from the action of $G = PSL(2,p)$ on the set $\mathscr{H}$ of right cosets of $H = D_{p-1}$ can be described in the following way. (For further details as well as all the proofs see [16].)

Firstly, note that in order for $q = \frac{p+1}{2}$ to be a prime we must have $p \equiv 1 \,(mod\, 4)$. Let $F = GF(p)$ and $F^* = F \setminus \{0\}$. For simplicity reasons we refer to the elements of G as matrices: this should cause no confusion. Let H consist of all the matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -x \\ x^{-1} & 0 \end{bmatrix} (x \in F^*).$$

For a typical element $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of G we let $\xi(g) = ad$ and $\eta(g) = a^{-1}b$. Further we let $\chi(g) = (\xi(g), \eta(g))$ be the *character* of g. Let $\sim$ be the equivalence relation on $F \times F^*$ defined by $(\xi, \eta) \sim (1-\xi, \frac{\xi\eta}{\xi-1})$ for $\xi \neq 0, 1$. There is then a natural identification of the sets $\mathscr{H}$ and $(F \times F^*)/_\sim \cup \{\infty\}$ where ∞ corresponds to H and (ξ, η) corresponds to the coset Hg satisfying $\chi(g) = (\xi, \eta)$.

Let S^* denote the set of all non-zero squares in F and let $N^* = F^* \setminus S^*$. For each $\xi \in S^*$ define the following subsets of $\mathscr{H}$. Let $\mathscr{S}_\xi^+ = \{(\xi, \eta) : \eta \in S^*\}$, $\mathscr{S}_\xi^- = \{(\xi, \eta) : \eta \in N^*\}$ and $\mathscr{S}_\xi = \mathscr{S}_\xi^+ \cup \mathscr{S}_\xi^-$. We observe that, for $\xi \neq 0, 1$, the sets $\{\mathscr{S}_\xi^+, \mathscr{S}_\xi^-\}$ and $\{\mathscr{S}_{1-\xi}^+, \mathscr{S}_{1-\xi}^-\}$ coincide. Moreover, since $(\frac{1}{2}, \eta) \sim (\frac{1}{2}, -\eta)$, it follows that the cardinality of $\mathscr{S}_\xi$ is $p-1$ except for $\xi = \frac{1}{2}$ when the cardinality is $\frac{p-1}{2}$. Similarly, the cardinalities of $\mathscr{S}_\xi^+$ and $\mathscr{S}_\xi^-$ are $\frac{p-1}{2}$ except for $\xi = \frac{1}{2}$ when the cardinalities are $\frac{p-1}{4}$.

The following result determines the suborbits of the action of G on $\mathscr{H}$.

Theorem 4.1. [16, Theorem] *The action of G on $\mathscr{H}$ has*

(i) $\frac{p+7}{4}$ *suborbits of length* $p-1$, *all of them self-paired. These are* $\mathcal{S}_0^+ \cup \mathcal{S}_1^+$, $\mathcal{S}_0^- \cup \mathcal{S}_1^-$ *and* $\mathcal{S}_\xi$ *for all those* ξ *which satisfy* $\xi^{-1}-1 \in N^*$.
(ii) $\frac{p-5}{2}$ *suborbits of length* $\frac{p-1}{2}$, *namely* $\mathcal{S}_\xi^+$ *and* $\mathcal{S}_\xi^-$ *where* $\xi^{-1}-1 \in S^*$. *Among them the self-paired suborbits correspond to all those* ξ *for which both* ξ *and* $\xi-1$ *belong to* N^* *and so their number is* $\frac{p-9}{4}$ *if* $p \equiv 1 \,(mod 8)$ *and* $\frac{p-5}{4}$ *if* $p \equiv 5 \,(mod 8)$.
(iii) *2 suborbits of length* $\frac{p-1}{4}$, *namely* $\mathcal{S}_{\frac{1}{2}}^+$ *and* $\mathcal{S}_{\frac{1}{2}}^-$ *which are self-paired if and only if* $p \equiv 1 \,(mod 8)$. ∎

It was noted in the proof of Theorem 2.1 that the rank 3 action of A_{p+1} on pairs from $\{1,2,...,p+1\}$ contains the action of $PSL(2,p)$ on cosets of D_{p-1}. Therefore the two orbital graphs can be found among the generalized orbital graphs $X(G,H,\mathcal{W})$, where $\mathcal{W}$ is a self-paired union of certain suborbits described in Theorem 4.1. In fact, it can be seen that the two orbital graphs are $X(G,H,\mathcal{S}_0 \cup \mathcal{S}_1)$ and its complement.

With the explicit description of the suborbits of G on $\mathcal{H}$ the construction of the relative generalized orbital graphs $X = X(G,H,\mathcal{W})$, where $\mathcal{W}$ is a self-paired union of suborbits of G, is relatively simple. Namely, the edge set of X is precisely the set $\{\{Hg, Hwg\} : g \in G, w \in \mathcal{W}\}$ and hence in order to determine the structure of X a rule for the multiplication of characters is applied.

The description of these graphs is best done via a factorization modulo the Sylow p-subgroup generated by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. For example, the smallest admissible pair of primes $p=13$ and $q=7$ gives rise to the following suborbits: $\mathcal{S}_0^+ \cup \mathcal{S}_1^+$, $\mathcal{S}_0^- \cup \mathcal{S}_1^-$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_5$ of size 12 and $\mathcal{S}_4^+$, $\mathcal{S}_4^-$ of size 6, all of them self-paired, and $\mathcal{S}_6^+$, $\mathcal{S}_6^-$ of size 6 and $\mathcal{S}_7^+$, $\mathcal{S}_7^-$ of size 3 which are not self-paired. Therefore each of the corresponding generalized orbital graphs is a union of the graphs $X(G,H,\mathcal{W})$ with $\mathcal{W} \in \{\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_5, \mathcal{S}_6, \mathcal{S}_4^+, \mathcal{S}_4^-, \mathcal{S}_7,\}$. As an illustration, using the notation of Frucht [6], the generalized orbital graph corresponding to $\mathcal{W} = \mathcal{S}_7$ is given below in Figure 4.

Finally, the subdegrees of the action of $G = PSL(2,61)$ on the set $\mathcal{A}$ of cosets of A_5 can be computed by the method of [9]. But we refer the reader also to [18], where a criterion for self-pairedness of suborbits is given. We obtain the following result.

Theorem 4.2. *The action of* $G = PSL(2,61)$ *on the set* $\mathcal{A}$ *of cosets of* A_5 *gives rise to:*
(i) *1 suborbit of size 1, 6 and 10 respectively;*
(ii) *2 suborbits of size 12;*
(iii) *4 suborbits of size 20;*
(iv) *5 suborbits of size 30;*
(v) *27 suborbits of size 60.*
Except for 8 suborbits of size 60, all other suborbits are self-paired. ∎

A description of the relative generalized orbital graphs is possible with methods mentioned before the statement of the above theorem. Namely, the stabilizers of suborbits of length 1, 6, 10, 12, 20, 30 and 60 are respectively A_5, D_{10}, D_6, Z_5, Z_3, Z_2 and 1. So we choose a particular subgroup A_5 inside $PSL(2,61)$ – for example,

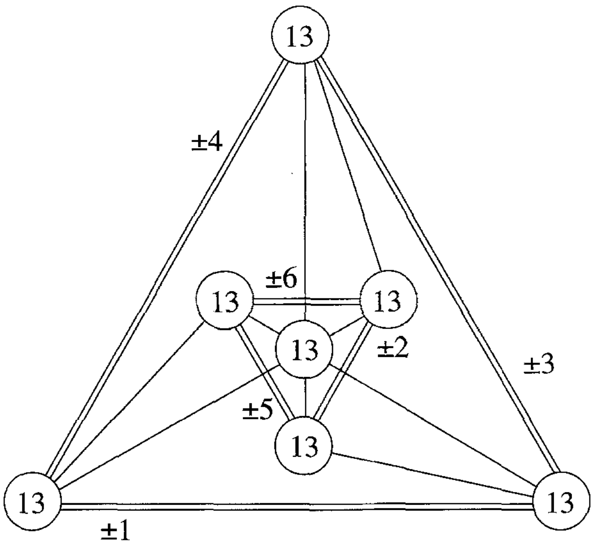

Fig. 4. $X(PSL(2,13), D_{12}, \mathcal{S}_7)$

the one generated by elements $\alpha = \begin{bmatrix} 3 & 0 \\ 0 & -20 \end{bmatrix}$ and $\beta = \begin{bmatrix} 8 & 22 \\ 22 & -8 \end{bmatrix}$ – as well as copies of the corresponding stabilizers and perform the necessary computations to obtain a more detailed information about the suborbits and the generalized orbital graphs.

5. Action of $PSL(2,q^2)$ on cosets of $PGL(2,q)$

The graphs arising from the action of $G = PSL(2,q^2)$ on the set $\mathcal{H}$ of right cosets of $H = PGL(2,q)$ can be described as follows. (For further details as well as all the proofs see [17]).

Let $F = GF(q)$. Fixing a non-square $\beta \in F$, let α be a solution of the equation $x^2 = \beta$. The extension $V = F(\alpha)$ of F is then isomorphic to $GF(q^2)$. We may let $G = PSL(2,V)$ and $H = <K, \tilde{a}>$, where $K = PSL(2,F)$ and $\tilde{a} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$. It can be shown that H is isomorphic to $PGL(2,F)$. For simplicity reasons we refer to the elements of G as matrices.

Let Ω be the set of all elements $x \in F$ such that $x = y^2\beta + y$ for some $y \in F$. Let Ω^+ and Ω^- be the sets of all non-zero squares and non-squares in Ω respectively. Every coset $Hg \neq H$ in $\mathcal{H}$ contains a canonical representative $\begin{bmatrix} 1 & y\alpha \\ z\alpha & 1 + yz\beta \end{bmatrix}$, $(y \in F, z \in F^*)$. Let us call $\omega = y^2 z^2 \beta + yz \in \Omega$ the *character* of Hg. It can be shown that this concept is well-defined. Moreover, one can then also see that each suborbit contains cosets with equal characters. More precisely, letting $\mathcal{T}_\omega$ be the set of all cosets Hg whose characters are equal to a given $\omega \in \Omega$, the suborbits of the action of G on $\mathcal{H}$ are determined by the following result.

Theorem 5.1. (Theorem 1.1, [17]) *The action of G on $\mathcal{H}$ has the following non-trivial suborbits.*
(i) $\mathcal{T}_0$ *is the only suborbit of length* (q^2-1).
(ii) $\mathcal{T}_{-(4\beta)^{-1}}$ *is the only suborbit of length* $q(q-1)/2$ *if* $q \equiv 1\,(mod 4)$ *and of length* $q(q+1)/2$ *if* $q \equiv 3\,(mod 4)$.
(iii) *suborbits* $\mathcal{T}_\omega$ $(\omega \in \Omega^- \backslash \{(-4\beta)^{-1}\})$ *have length* $q(q-1)$; *there number is* $(q-5)/4$ *if* $q \equiv 1\,(mod 4)$ *and* $(q-3)/4$ *if* $q \equiv 3\,(mod 4)$.
(iv) *suborbits* $\mathcal{T}_\omega$ $(\omega \in \Omega^+ \backslash \{(-4\beta)^{-1}\})$ *have length* $q(q+1)$; *there number is* $(q-1)/4$ *if* $q \equiv 1\,(mod 4)$ *and* $(q-3)/4$ *if* $q \equiv 3\,(mod 4)$.
Furthermore, all suborbits are self-paired. ∎

Let us mention that the lengths of these suborbits can also be deduced from formula (7) of [5]. Moreover, an alternative approach is given in [20].

With the explicit description of the suborbits of G on $\mathcal{H}$ the construction of the corresponding generalized orbital graphs $X(G,H,\mathcal{W})$, where $\mathcal{W}$ is a union of suborbits, is relatively simple. The structure of these graphs is best understood via a factorization modulo a cyclic group of order $\frac{q^2+1}{2}$. The computations however are rather tedious. To illustrate the ideas we thus choose to consider the smallest case $q=3$ which gives rise to graphs $L(K_6)$ and $L(K_6^c)$, the Fermat graphs arising also in rows 2, 7 of Table 1.

The two non-trivial H-suborbits are $\mathcal{T}_1$ of length 6 containing cosets with representatives

$$\begin{bmatrix} 1+\alpha & 0 \\ 0 & -1+\alpha \end{bmatrix}, \begin{bmatrix} -1+\alpha & 1 \\ 1-\alpha & \alpha \end{bmatrix}, \begin{bmatrix} -1-\alpha & 1 \\ 1+\alpha & -\alpha \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1-\alpha \\ \alpha & -1+\alpha \end{bmatrix}, \begin{bmatrix} 1 & -1-\alpha \\ \alpha & -1-\alpha \end{bmatrix}, \begin{bmatrix} 1 & -\alpha \\ \alpha & -1 \end{bmatrix}$$

and $\mathcal{T}_0$ of length 8 containing cosets with representatives

$$\begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ \alpha & 1+\alpha \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ \alpha & 1-\alpha \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -\alpha & 1-\alpha \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -\alpha & 1+\alpha \end{bmatrix}.$$

Two complementary graphs having 15 vertices are obtained. The graph $X(G,\mathcal{T}_1)= K_6^c$ is depicted in Figure 2, where the notation of Frucht [6] is used to emphasize the three orbits of the cyclic group of order 5 generated by the element $\begin{bmatrix} 1 & -1 \\ \alpha & 1-\alpha \end{bmatrix}$.

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